A Note on the Eigenvalues of Real Symmetric Matrices BB^T

Yin,Daijun*

College of Mathematics and Science, Xinjiang Education Institute, Urumqi, Xinjiang, 830063, China

Abstract: We discuss the eigenvalues of real symmetric matrices, especially focus on the eigenvalues of matrix BB^T where B expresses the adjacency relation between two parts of a semiregular bipartite graph. Based on this, the existence of two kinds of (r, k)-semiregular bipartite graphs are excluded.

Keywords: Real symmetric matrices; Eigenvalues; Semiregular bipartite graph

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1. Preliminaries

Let $G=(V(G),E(G))$ be undirected simple graph on n vertices with adjacency matrix A = A(G). Denote by $\lambda_1, \lambda_2, \cdots, \lambda_t$ all the distinct eigenvalues of A with multiplicities $m_1, m_2, \cdots, m_t$ $(\sum_{i=1}^{t} m_i = n)$, respectively.

These eigenvalues are also called the eigenvalues of G. All the eigenvalues together with their multiplicities are called the spectrum of G denoted by Spec(G) $=\{[\lambda_1]^{m_1}, [\lambda_2]^{m_2}, \ldots, [\lambda_t]^{m_t}\}$[1].

Bipartite graphs with few distinct eigenvalues have nice combinatorial properties and many studies on it are somehow relevant to the combinational designs, such as [4, 5, 6]. Most of the studies focus on the spectral characterization of regular or semiregular bipartite graphs with five distinct eigenvalues [2, 3]. Here we discuss the non-existence of some semiregular bipartite graphs with five distinct eigenvalues in terms of the properties of real symmetric matrices BB^T.

Lemma 1.1. *Let M be a symmetric matrix of size n, having constant diagonal and constant row sums ω, and spectrum* $\{\omega^1, \sigma^1, 0^{n-2}\}$, *with ω = 0; then n is even and (possibly after permuting rows and columns) M can be written as*

$$M=\begin{pmatrix} \frac{\omega+\sigma}{n}J_{\frac{n}{2}} & \frac{\omega-\sigma}{n}J_{\frac{n}{2}} \\ \frac{\omega+\sigma}{n}J_{\frac{n}{2}} & \frac{\omega-\sigma}{n}J_{\frac{n}{2}} \end{pmatrix}.$$

Lemma 1.2. *Let M be a real symmetric matrix of size n, with spectrum* $\{\sigma^m, 0^{n-m}\}$ *where m ≥ 1. Then* $M=\sigma(u_1u_1^T+\cdots+u_mu_m^T)$ *with* $u_1, \cdots, u_m$ *being the pairwise orthogonal eigenvectors of M with respect to σ.*

2. Main Results

Theorem 2.1. *Connected (r, k)-semiregular bipartite graph with size* n_1, n_2 *of each part and spectrum*

$$\{\sqrt{rk},\ \lambda_2^{n_1-2},\ 0^{n_2-n_1+2},\ -\lambda_2^{n_1-2},\ -\sqrt{rk}\}$$

does not exist.

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Corresponding Author

Yin,Daijun (1981-), female, is an associate professor and a doctor of science at Xinjiang Normal College (Xinjiang Education Institute). Her major is applied mathematics and her research direction is algebraic graph theory.

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Proof. Suppose that G is a (r, k)-semiregular bipartite graph with adjacency matrix $A = \begin{pmatrix} O & B \\ B^T & O \end{pmatrix}$. From the spectrum of G we have

$$Spec(BB^T) = \{[rk]^1, [\lambda_2^2]^{n_1-2}, [0]^1\}.$$

Further,

$$Spec(BB^T - \lambda_2^2 I) := \{[rk - \lambda_2^2]^1, [-\lambda_2^2]^1, [0]^{n_1-2}\}.$$

Recall that $BB^T - \lambda_2^2 I$ is a symmetric matrix of size n_1, having constant diagonal $r - \lambda_2^2$ and constant row sums $rk - \lambda_2^2$. From lemma 1.1 we have

$$BB^T - \lambda_2^2 I = \begin{pmatrix} \frac{rk-2\lambda_2^2}{n_1} J_{\frac{n_1}{2}} & \frac{rk}{n_1} J_{\frac{n_1}{2}} \\ \frac{rk}{n_1} J_{\frac{n_1}{2}} & \frac{rk-2\lambda_2^2}{n_1} J_{\frac{n_1}{2}} \end{pmatrix}. \quad (1)$$

On the other hand, by taking $f(x) = x(x^2 - \lambda_2^2)$ and from the spectral decomposition of $f(A)$ we get

$$A(A^2 - \lambda_2^2 I) = \rho(\rho^2 - \lambda_2^2)P_1 - \rho(\rho^2 - \lambda_2^2)P_2$$

$$= \frac{\rho(\rho^2-\lambda_2^2)}{n_1+\frac{k}{r}n_2} \begin{pmatrix} J_{n_1} & \sqrt{\frac{k}{r}}J \\ \sqrt{\frac{k}{r}}J & \frac{k}{r}J_{n_2} \end{pmatrix} - \frac{\rho(\rho^2-\lambda_2^2)}{n_1+\frac{k}{r}n_2} \begin{pmatrix} J_{n_1} & -\sqrt{\frac{k}{r}}J \\ -\sqrt{\frac{k}{r}}J & \frac{k}{r}J_{n_2} \end{pmatrix}, \quad (2)$$

and then

$$B(B^TB - \lambda_2^2 I) = (BB^T - \lambda_2^2 I)B = \frac{2\rho(\rho^2 - \lambda_2^2)}{n_1 + \frac{k}{r}n_2}\sqrt{\frac{k}{r}}J = \frac{k(rk - \lambda_2^2)}{n_1}J. \quad (3)$$

We partition V_1 into two equal parts $V_{11} \cup V_{12}$ and correspondingly denote $B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$ such that each B_i expresses the adjacency relation from V_{1i} to V_2 for $i = 1, 2$. From (1) and (3) we have

$$\begin{cases} \frac{rk-2\lambda_2^2}{n_1} JB_1 + \frac{rk}{n_1} JB_2 = \frac{k(rk-\lambda_2^2)}{n_1} J_{\frac{n_1}{2}\times n_2} & \text{(i)}. \\ \frac{rk}{n_1} JB_1 + \frac{rk-2\lambda_2^2}{n_1} JB_2 = \frac{k(rk-\lambda_2^2)}{n_1} J_{\frac{n_1}{2}\times n_2} & \text{(ii)}. \end{cases}$$

Assume that

$$JB_1 = \mathbf{e}_{\frac{n_1}{2}} \otimes (p_1, p_2, \cdots, p_{n_2}), \quad JB_2 = \mathbf{e}_{\frac{n_1}{2}} \otimes (q_1, q_2, \cdots, q_{n_2}),$$

where p_i, q_i, respectively, denote the number of vertices in V_{11} and V_{12} that adjacent to $\omega_i \in V_2$, for $i = 1, \cdots, n_2$. Clearly, $p_i + q_i = k$. We will give following three claims.

Claim 1: For any $i \in \{1, \cdots, n_2\}$, $p_i \neq 0$ and $q_i \neq 0$.

If $p_i = 0$, by comparing the ith column of both sides of equality (i) we have

$$\frac{rk}{n_1} q_i = \frac{k(rk - \lambda_2^2)}{n_1}.$$

Note that $\lambda_2^2 = \frac{n_1 r - rk}{n_1 - 2}$, then $q_i = k - \frac{n_1 - k}{n_1 - 2}$. Since q_i is an integer and $k \geq 2$, it forces to have $k = 2$, then $1 = q_i = k = 2$ which is a contradiction. If $q_i = 0$, we similarly deduce a contradiction from equality (ii).

Claim 2: For any $i, j \in \{1, \cdots, n_2\}$, $p_i = p_j$ and $q_i = q_j$.

Suppose that there are $i, j \in \{1, \cdots, n_2\}$ such that $p_i \neq p_j$, and correspondingly, $q_i \neq q_j$ from (i) and (ii). Furthermore, from the *i*th and *j*th columns of equality (*i*) and (*ii*) we get

$$\frac{rk-2\lambda_2^2}{n_1}(p_i-p_j)=\frac{rk}{n_1}(q_j-q_i),\quad \frac{rk}{n_1}(p_i-p_j)=\frac{rk-2\lambda_2^2}{n_1}(q_j-q_i).$$

Then we obtain that

$$\frac{rk-2\lambda_2^2}{rk}=\frac{rk}{rk-2\lambda_2^2}\ \Rightarrow\ \lambda_2^2=rk.$$

A contradiction.

Claim 3: For any $i \in \{1,\cdots,n_2\}$, $p_i=p_j=\frac{k}{2}$.

It is known that $p_1=\cdots=p_{n_2}=p$ and $q_1=\cdots=q_{n_2}=q$ from claim 2. So we have $JB_1=pJ$, $JB_2=qJ$ and from (i), (ii)

$$\frac{rk-2\lambda_2^2}{n_1}pJ+\frac{rk}{n_1}qJ=\frac{rk}{n_1}pJ+\frac{rk-2\lambda_2^2}{n_1}qJ,\ \Rightarrow\ \frac{-2\lambda_2^2}{n_1}p=\frac{-2\lambda_2^2}{n_1}q.$$

Thus $p=q=\frac{k}{2}$ since $\lambda_2^2\neq 0$.

So far, we assert that both $G[V_{11}\cup V_2]$ and $G[V_{12}\cup V_2]$ are $(r,\frac{k}{2})$-semiregular bipartite graphs. On the other hand, from

$$\begin{pmatrix} B_1B_1^T & B_1B_2^T \\ B_2B_1^T & B_2B_2^T \end{pmatrix}=BB^T=\begin{pmatrix} \frac{rk-2\lambda_2^2}{n_1}J_{\frac{n_1}{2}}+\lambda_2^2I & \frac{rk}{n_1}J_{\frac{n_1}{2}} \\ \frac{rk}{n_1}J_{\frac{n_1}{2}} & \frac{rk-2\lambda_2^2}{n_1}J_{\frac{n_1}{2}}+\lambda_2^2I \end{pmatrix}$$

we get

$$B_1B_1^T=B_2B_2^T=\frac{rk-2\lambda_2^2}{n_1}J_{\frac{n_1}{2}}+\lambda_2^2I.$$

Clearly, λ_2^2 is integer because *rk* is integer and $\Phi_A(G;x)$ is a monic polynomial with integer coefficients. We might as well set $\frac{rk-2\lambda_2^2}{n_1}=\frac{r(k-2)}{n_1-2}=\lambda$, then $\lambda_2^2=r-\lambda$. By lemma, both B_1, B_2 are incidence matrix of $(\frac{n_1}{2},n_2,r,\frac{k}{2},\lambda)$-*BIBD*, and hence $G[V_{11}\cup V_2]$ and $G[V_{12}\cup V_2]$ are the incidence graph of this design. However, $B_1B_2^T=\frac{rk}{n_1}J$ means that any vertex of V_{11} share the same number of neighbours in V_2 with any vertex of V_{12}, which is impossible. It completes our proof.

Theorem 2.2. *Connected (r, k)-semiregular bipartite graph with size n_1, n_2 of each part and spectrum*

$$\{\sqrt{rk},\ \lambda_2^{n_1-3},\ 0^{n_2-n_1+4},\ -\lambda_2^{n_1-3},\ -\sqrt{rk}\}$$

does not exist.

Proof. Let *G* be a (*r, k*)-semiregular bipartite graph with spectrum $\{\sqrt{rk},\ \lambda_2^{n_1-3},\ 0^{n_2-n_1+4},\ -\lambda_2^{n_1-3},\ -\sqrt{rk}\}$. Then BB^T has spectrum $\{[rk]^1,[\lambda_2^2]^{n_1-3},[0]^2\}$. Furthermore, $Spec(BB^T-\lambda_2^2I)=\{[rk-\lambda_2^2]^1,[-\lambda_2^2]^2,[0]^{n_1-3}\}$.

Let $M=BB^T-\lambda_2^2I-\frac{rk-\lambda_2^2}{n_1}J$ be a real symmetric matrix of size n_1 with constant row sum 0 and constant diagonal $r-\lambda_2^2-\frac{rk-\lambda_2^2}{n_1}$, $Spec(M)=\{[-\lambda_2^2]^2,[0]^{n_1-2}\}$. Then from Lemma 1.2,

$$M=-\lambda_2^2(uu^T+ww^T)$$

where *u, w* are the orthogonal eigenvectors of *M* with respect to $-\lambda_2^2$ (and also of BB^T with respect to 0). Suppose that $u=[u_1,\cdots,u_{n_1}]^T,\ w=[w_1,\cdots,w_{n_1}]^T$. Note that $rk+(n_1-3)\lambda_2^2=Tr(BB^T)=n_1r$ so for any $i\in\{1,\cdots,n_1\}$ we have

$$\begin{cases} u_i\sum_{1\le l\le n_1}u_l+w_i\sum_{1\le l\le n_1}w_l=0, \\ u_i^2+w_i^2=\frac{1}{-\lambda_2^2}(r-\lambda_2^2-\frac{rk-\lambda_2^2}{n_1})=\frac{2}{n_1} \end{cases}. \quad (4)$$

It is easy to verify that

$$\begin{aligned} 0 &= \sum_{i=1}^{n_1}(u_i\sum_{1\le l\le n_1}u_l+w_i\sum_{1\le l\le n_1}w_l) \\ &= \sum_{i=1}^{n_1}u_i^2+\sum_{i=1}^{n_1}w_i^2+\sum_{i=1}^{n_1}u_i\sum_{l\ne i}u_l+\sum_{i=1}^{n_1}w_i\sum_{l\ne i}w_l \\ &= (u_1+u_2+\cdots+u_{n_1})^2+(w_1+w_2+\cdots+w_{n_1})^2. \end{aligned}$$

Thus $\sum_{i=1}^{n_1}u_i=0$ and $\sum_{i=1}^{n_1}w_i=0$, and hence $u_1^2=u_2^2=\cdots=u_{n_1}^2,\ \ w_1^2=w_2^2=\cdots=w_{n_1}^2$. It follows that there are exactly $\frac{n_1}{2}$'s u_i equal α and others equal -α, and there are exactly $\frac{n_1}{2}$'s w_i equal β and others equal -β. Also note that $u\perp w$, so it forces to have

$$u=(\alpha\mathbf{e}_{\frac{n_1}{2}}^T,-\alpha\mathbf{e}_{\frac{n_1}{2}}^T)^T,\ w=(\beta\mathbf{e}_{\frac{n_1}{4}}^T,-\beta\mathbf{e}_{\frac{n_1}{4}}^T,\beta\mathbf{e}_{\frac{n_1}{4}}^T,-\beta\mathbf{e}_{\frac{n_1}{4}}^T),$$

and then

$$M=-\lambda_2^2\begin{pmatrix} (\alpha^2+\beta^2)J & (\alpha^2-\beta^2)J & (-\alpha^2+\beta^2)J & (-\alpha^2-\beta^2)J \\ (\alpha^2-\beta^2)J & (\alpha^2+\beta^2)J & (-\alpha^2-\beta^2)J & (-\alpha^2+\beta^2)J \\ (-\alpha^2+\beta^2)J & (-\alpha^2-\beta^2)J & (\alpha^2+\beta^2)J & (\alpha^2-\beta^2)J \\ (-\alpha^2-\beta^2)J & (-\alpha^2+\beta^2)J & (\alpha^2-\beta^2)J & (\alpha^2+\beta^2)J \end{pmatrix}.$$

By simply calculating, we have

$$det(xI-M)=x^{n_1-2}(x+n_1\lambda_2^2\alpha^2)(x+n_1\lambda_2^2\beta^2),$$

and the nonzero eigenvalue of *M* is

$$-\lambda_2^2=-n_1\lambda_2^2\alpha^2=-n_1\lambda_2^2\beta^2.$$

Thus $\alpha^2=\beta^2=\frac{1}{n_1}$. Recall that $M=BB^T-\lambda_2^2I-\frac{rk-\lambda_2^2}{n_1}J$, so we get

$$BB^T-\lambda_2^2I=M+\frac{rk-\lambda_2^2}{n_1}J=\begin{pmatrix} \frac{rk-3\lambda_2^2}{n_1}J & \frac{rk-\lambda_2^2}{n_1}J & \frac{rk-\lambda_2^2}{n_1}J & \frac{rk+\lambda_2^2}{n_1}J \\ \frac{rk-\lambda_2^2}{n_1}J & \frac{rk-3\lambda_2^2}{n_1}J & \frac{rk+\lambda_2^2}{n_1}J & \frac{rk-\lambda_2^2}{n_1}J \\ \frac{rk-\lambda_2^2}{n_1}J & \frac{rk+\lambda_2^2}{n_1}J & \frac{rk-3\lambda_2^2}{n_1}J & \frac{rk-\lambda_2^2}{n_1}J \\ \frac{rk+\lambda_2^2}{n_1}J & \frac{rk-\lambda_2^2}{n_1}J & \frac{rk-\lambda_2^2}{n_1}J & \frac{rk-3\lambda_2^2}{n_1}J \end{pmatrix}. \quad (5)$$

Now we partition V_1 into four equal parts $V_{11}\cup V_{12}\cup V_{13}\cup V_{14}$ and correspondingly denote $B=(B_1^T,B_2^T,B_3^T,B_4^T)^T$ such that each B_i expresses the adjacency relation from V_{1i} to V_2 for *i*=1,2,3,4. On the other hand, from the demonstration of theorem we also obtain equality (3). Combining (5) we get following four equalities.

$$\begin{cases} \frac{rk-3\lambda_2^2}{n_1}JB_1 + \frac{rk-\lambda_2^2}{n_1}JB_2 + \frac{rk-\lambda_2^2}{n_1}JB_3 + \frac{rk+\lambda_2^2}{n_1}JB_4 & \text{(a);} \\ \frac{rk-\lambda_2^2}{n_1}JB_1 + \frac{rk-3\lambda_2^2}{n_1}JB_2 + \frac{rk+\lambda_2^2}{n_1}JB_3 + \frac{rk-\lambda_2^2}{n_1}JB_4 & \text{(b);} \\ \frac{rk-\lambda_2^2}{n_1}JB_1 + \frac{rk+\lambda_2^2}{n_1}JB_2 + \frac{rk-3\lambda_2^2}{n_1}JB_3 + \frac{rk-\lambda_2^2}{n_1}JB_4 & \text{(c);} \\ \frac{rk+\lambda_2^2}{n_1}JB_1 + \frac{rk-\lambda_2^2}{n_1}JB_2 + \frac{rk-\lambda_2^2}{n_1}JB_3 + \frac{rk-3\lambda_2^2}{n_1}JB_4 & \text{(d).} \end{cases}$$

As parallel as the proof of theorem , we can also obtain three claims and finally get, for *i*=1,2,3,4, $JB_i = \frac{k}{4}J$, each $G[V_{1i} \cup V_2]$ is a $(r,\frac{k}{4})-$ semiregular bipartite graph, and furthermore,

$$B_1B_1^T = B_2B_2^T = B_3B_3^T = B_4B_4^T = \frac{rk-3\lambda_2^2}{n_1}J + \lambda_2^2 I.$$

Suppose that $\frac{rk-3\lambda_2^2}{n_1} = \lambda$, it is easy to verify that $r - \lambda = \lambda_2^2$. So each $G[V_{1i} \cup V_2]$ is an incidence graph of $(\frac{n_1}{4}, n_2, r, \frac{k}{4}, \lambda)$-*BIBD*. Thus a similar contradiction can be deduced from $B_1B_j^T = \frac{rk-\lambda_2^2}{n_1}J$ for *j*=2,3 and $B_1B_4^T = \frac{rk+\lambda_2^2}{n_1}J$.

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